Final exam probability probability theory (WIKR-06)

18 June 2020, 8.30 – 12.00

- Work on this exam if the **digit sum** of your student id is **odd**.
- Before the start of the exam, everybody taking part in the exam must sign the student declaration in the exam environment.
- To check for possible fraud, an unannounced sample of students will be contacted soon after the exam.
- The answers need to be written by hand, scanned and submitted within the time limit. You must upload your exam in a single pdf file.
- Every exercise needs to be handed in on a separate sheet.
- Write your name and student number **on every sheet**.
- It is forbidden to communicate with other persons during the exam, except with the course instructor.
- The only tools and aids that you are allowed to use are a non-programmable calculator (not a phone!), and the following material from the nestor course environment:
  a) The pdf file of the lecture notes (not videos, not scribbles).
  b) The pdf files of the tutorial problems.
  c) The pdf files of the homework problems.
  d) The pdf files of the solutions to the homework problems.
- Always give a short proof of your answer or a calculation to justify it, or clearly state the facts from the lecture notes or homework you are using.
- Simplify your final answers as much as possible.
- **NOTA BENE.** Using separate sheets for the different exercises, solving the exam corresponding to your student number, writing your name and student number on all sheets, and submitting all sheets in a single pdf is worth 10 out of the 100 points.

Problem 1 (a:4, b:6, c:6, d:4, e:4, f:6 pts).
The joint pdf of a random vector (X,Y) equals

$$f_{X,Y}(x,y) := \begin{cases} cx^{-5} & \text{if } 0 < y < x \text{ and } 3 < x, \\ 0 & \text{otherwise.} \end{cases}$$

a) Determine c.

b) Determine $f_X(x)$ and $f_Y(y)$.

c) Determine $f_{Y|X}(y|x)$ for $x > 3$ and $f_{X|Y}(x|y)$ for $y > 0$.

d) Are X and Y independent?

e) Compute $\mathbb{E}[Y|X=7]$ and $\mathsf{Var}(Y|X=7)$.

f) Compute $\mathbb{E}[X|Y=7]$ and $\mathsf{Var}(X|Y=7)$.

Note. If you could not solve part b), then you may assume (incorrectly) for parts c) and d) that $f_Y(y) = y^{-4}1_{[1,\infty)}(y)$ and $f_X(x) = x^{-4}1_{[3,\infty)}(x)$.
Note. If you could not solve part d), then you may assume (incorrectly) for parts e) and f) that $f_{X|Y}(x|y) = 6y^{-12}x^{-7}1_{[y^2,\infty)}(x)$ and $f_{Y|X}(y|x) = \frac{1}{x^2}1_{[0,x^2]}(y)$.

Problem 2 (a:8, b:2, c:4, d:6, e:6, f:4 pts).
The MANROBA bank launches its own cryptocurrency, ROBACOIN. To encourage customers to pay in this currency, it designs new Betalpassen. MANROBA generates independent random variables $(\Xi_i)_{i\geq 1}$ where the pdf of each Ξ_i is

$$f(x) = 4x^3.$$

Independently of $(\Xi_i)_{i\geq 1}$, MANROBA also generates independent random variables $(V_j)_{j\geq 1}$, where each V_j is uniform on $[0,1]$. Now, the ith customer receives Betalpas with parameter Ξ_i. The jth time when customer i pays in ROBACOIN, this customer receives a bonus of 1 Euro if $V_j \leq \Xi_i$ and no bonus otherwise.

a) Let $T_1 = \#\{j \leq 2 : V_j \leq \Xi_1\}$ be the total bonus received by customer 1 in the first 2 transactions. Compute the joint cdf $F_{\Xi_1,T_1}(x,y)$ for $x \in [0,1]$ and $y \in \{0,1,2\}$.

When ROBACOIN is rolled out, an intern accidentally deleted the parameters $(\Xi_i)_{i\geq 1}$ from the database.

b) Compute the expected bonus per ROBACOIN transaction if the parameter Ξ_i is unknown. That is, compute $\mathbb{E}[1\{V_j \leq \Xi_i\}]$ for $i,j \geq 1$.

The MANROBA bank expands its portfolio to include risky assets of type C and D. Within one month, asset C generates a profit H_C with cdf

$$F_{H_C}(y) = 1 - e^{-2y} - 2ye^{-2y}.$$

Within one month, asset D generates independently a profit H_D with cdf

$$F_{H_D}(x) = 1 - e^{-3x} - 3xe^{-3x}.$$

c) Compute the joint pdf of H_C and H_D.

The bank sells a product to their customers that combines two assets of type C with three assets of type D.

d) Compute the pdf of the portion $2H_C/(2H_C + 3H_D)$ that the two assets of type C contribute to the total profit.

The MANROBA bank manages a different portfolio consisting of two categories of risky assets: 800 assets of type A and 200 assets of type B. Within one month, type A generates no profit with probability 75 %, a profit of 1 with probability 20 % and a profit of 2 with probability 5 %. Within one month, type B generates no profit with probability 80 %, a profit of 1 with probability 15 % and a profit of 2 with probability 5 %. Inadvertently, an intern crashes the MANROBA database, so that the information which asset belongs to which category is lost. The intern is sent back to school and a new intern is hired, who should recover the information from the data of the last 9 month.

e) The new intern picks one of the assets from the database at random and finds that it generated a profit in 3 of the last 9 months. Compute the probability that this asset is of type A. Compute the expected profit of this asset in the next month.

f) Compute the mgf of an asset of type A. Compute the mgf of an asset of type B.

Problem 3 (a:6, b:4, c:6, d:10, e:4 pts).
Niklaus, Daniel, Jacob and Johann visit a chocolate factory. The length of a chocolate bar drawn at random is known to follow a random variable N with pdf

$$f_N(x) = 8x^2 e^{-4x}.$$

Niklaus picks a bar at random and shares it among Daniel, Jacob and Johann. Therefore, independently of N, he chooses two random subdivision points V and V' that are independent and uniform on $[0,1]$. Then, Daniel receives $S_1 = N(1-\max\{V,V'\})$, Jacob receives $S_2 = N(\max\{V,V'\} - \min\{V,V'\})$ and Johann receives $S_3 = N\min\{V,V'\}$.

a) Compute the joint cdf and pdf of $\max\{V,V'\}$ and $\min\{V,V'\}$.

b) Compute the joint pdf of N and $\min\{V,V'\}$.

c) Show that S_1 and S_3 both have the pdf $h(x) = 4e^{-4x}$.

d) Compute the joint cdf of S_1, S_3. Are S_1 and S_3 independent?

e) Show that $\mathbb{E}[S_1] = \mathbb{E}[S_2] = \mathbb{E}[S_3] = 1/4$.

Solutions
a)

$$\int_0^\infty \int_0^\infty x^{-5} 1_{[3,\infty)\times[0,x]}(x,y)\mathrm{d}y\mathrm{d}x = \int_3^\infty x^{-4}\mathrm{d}x = \frac{1}{81}.$$

Hence, $c = 81$.

b) For $y > 0$,

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\mathrm{d}x = 81\int_{3\vee y}^\infty x^{-5}\mathrm{d}x = \frac{81}{4}(3\vee y)^{-4}.$$

For $x > 3$,

$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)\mathrm{d}y = 81x^{-5}\int_0^x 1\mathrm{d}y = 81x^{-4}.$$

c)

$$f_X(x)f_Y(y) = \frac{81^2}{4}(3\vee y)^{-4}1_{(0,\infty)}(y)x^{-4}1_{[3,\infty)}(x) \neq f_{X,Y}(x,y).$$

Hence, X and Y are not independent.
d)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{x}1_{[0,x]}(y).$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = 4(3\vee y)^4 x^{-5} 1_{[3\vee y,\infty)}(x).$$

e) By part d), when conditioned on $X = 7$, then Y is uniform on $[0,7]$. Hence, $\mathbb{E}[Y|X = 7] = (0+7)/2 = 3/2$ and $\mathsf{Var}(Y|X = 7) = 7^2/12 = 49/12$.
f) When conditioned on $Y = 7$, then X is Pareto with parameter 4 and minimum 7. Hence,

$$\mathbb{E}[X\,|\,Y = 7] = \int_7^\infty x f_{X|Y}(x|7)\mathrm{d}x = \int_7^\infty 4\cdot 7^4 x^{-4}\mathrm{d}x = 28/3.$$

$$\mathbb{E}[X^2\,|\,Y = 7] = \int_7^\infty x^2 f_{X|Y}(x|7)\mathrm{d}x = \int_7^\infty 4\cdot 7^4 x^{-3}\mathrm{d}x = 98.$$

Hence, $\mathsf{Var}(X|Y = 7) = 98-(28/3)^2 = 98/9$.

Solutions
a) For $x \in \{0,1,2\}$ and $y \in [0,1]$ the joint cdf equals

$$\begin{aligned}
F_{T_1,\Xi_1}(x,y) &= \mathbb{P}(T_1 \le x, \Xi_1 \le y)\\
&= \sum_{k\le x}\mathbb{P}(T_1 = k, \Xi_1 \le y)\\
&= \sum_{k\le x}\sum_{\substack{I\subset\{1,2\}\\ \#I=k}}\mathbb{P}(V_i \le \Xi_1 \text{ for } i \in I, V_{i'} > \Xi_1 \text{ for } i' \notin I, \Xi_1 \le y)\\
&= \sum_{k\le x}\sum_{\substack{I\subset\{1,2\}\\ \#I=k}} 4\int_0^y \mathbb{P}(V_i \le s \text{ for } i \in I, V_{i'} > s \text{ for } i' \notin I)s^3\mathrm{d}s\\
&= \sum_{k\le x} 4\binom{2}{k}\int_0^y s^{k+3}(1-s)^{2-k}\mathrm{d}s.
\end{aligned}$$

Hence, for $x \in \{0,1,2\}$ and $y \in [0,1]$,

$$\begin{aligned}
F_{T_1,\Xi_1}(x,y) &= 4\big(\tfrac14 y^4 - \tfrac25 y^5 + \tfrac16 y^6\big) + 8\big(\tfrac15 y^5 - \tfrac16 y^6\big)1_{\{1,2\}}(x) + \tfrac23 y^6\cdot 1_{\{2\}}(x)\\
&= \begin{cases} y^4 - \frac85 y^5 + \frac23 y^6 & \text{if } x = 0,\\ y^4 - \frac23 y^6 & \text{if } x = 1,\\ y^4 & \text{if } x = 2.\end{cases}
\end{aligned}$$

b) Since V_j and Ξ_i are independent,

$$\mathbb{E}[1\{V_j \le \Xi_i\}] = 4\int_0^1\int_0^1 1_{[0,\theta]}(u)\theta^3\mathrm{d}u\mathrm{d}\theta = 4\int_0^1 \theta^4\mathrm{d}\theta = \frac45.$$

c) The pdf of H_C and H_D equal

$$f_{H_C}(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_{H_C}(x) = 4xe^{-2x} \qquad \text{and} \qquad f_{H_D}(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_{H_D}(y) = 9ye^{-3y}.$$

Since H_C and H_D are independent, the joint pdf equals $f_{H_C,H_D}(x,y) = f_{H_C}(x)f_{H_D}(y) = 36xye^{-2x-3y}$.
d) We first compute the cdf

$$\begin{aligned}
\mathbb{P}(2H_C/(2H_C+3H_D) \le r) &= 36\int_0^\infty\int_0^\infty 1\{2x \le r(2x+3y)\}xe^{-2x}ye^{-3y}\mathrm{d}x\mathrm{d}y\\
&= \int_0^\infty e^{-z}\int_0^{rz} x(z-x)\mathrm{d}x\mathrm{d}z\\
&= \Big(\int_0^\infty e^{-z}z^3\mathrm{d}z\Big)\cdot\int_0^r v(1-v)\mathrm{d}v\\
&= 6\int_0^r v(1-v)\mathrm{d}v.
\end{aligned}$$

Computing the derivative with respect to r gives that $f_{2H_C/(2H_C+3H_D)}(r) = 6r(1-r)$.

e) We use Bayes' formula. Let E_A be the event that the asset is of type A, E_B the event that the asset is of type B and E_0 be the event that the asset generates a profit in 3 of the last 9 months. Then,

$$\mathbb{P}(E_A|E_0) = \frac{\mathbb{P}(E_0|E_A)\mathbb{P}(E_A)}{\mathbb{P}(E_0|E_A)\mathbb{P}(E_A)+\mathbb{P}(E_0|E_B)\mathbb{P}(E_B)} = \frac{\binom{9}{3}0.75^6\cdot 0.25^3\cdot 0.8}{\binom{9}{3}0.75^6\cdot 0.25^3\cdot 0.8+\binom{9}{3}0.8^6\cdot 0.2^3\cdot 0.2} \approx 84.1\%.$$

Expected profit of type A: $0.2\cdot 1+0.05\cdot 2 = 0.3$. Expected profit of type B: $0.15\cdot 1+0.05\cdot 2 = 0.25$. Hence, the expected profit of the selected asset equals approximately $0.3\cdot 0.841+0.25\cdot(1-0.841) = 0.292$.
f) $M_A(z) = 0.75+0.2z+0.05z^2$ and $M_B(z) = 0.8+0.1z+0.1z^2$.

Solutions
a) Let $x,y\in[0,1]$. First, if $x\le y$,

$$F_{V\wedge V',V\vee V'}(x,y) = \mathbb{P}(V\wedge V'\le x, V\vee V'\le y) = \mathbb{P}(V,V'\le y)-\mathbb{P}(x\le V,V'\le y) = y^2-(y-x)^2 = 2yx-x^2.$$

Now, if $x>y$,

$$F_{V\wedge V',V\vee V'}(x,y) = \mathbb{P}(V\wedge V'\le x, V\vee V'\le y) = \mathbb{P}(V,V'\le y) = y^2.$$

Hence,

$$f_{V\wedge V',V\vee V'}(x,y) = \begin{cases} 2 & \text{if } 0\le x\le y\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

b) Setting $y=1$ in a) and taking the derivative gives $f_{V\wedge V'}(x) = 2(1-x)$. Hence, by independence,

$$f_{V\wedge V',N}(x,y) = 16(1-x)y^2e^{-4y}.$$

c) The random variables $V\wedge V'$ and $1-V\vee V' = (1-V)\wedge(1-V')$ have the same distribution, so it suffices to show that S_1 is an exponential random variable

$$\begin{aligned}\mathbb{P}((V\wedge V')N>r) &= 16\int_0^\infty\int_0^1 1_{[r,\infty)}(xy)(1-x)y^2e^{-4y}\mathrm{d}x\mathrm{d}y\\ &= 8\int_r^\infty(1-r/y)^2y^2e^{-4y}\mathrm{d}y\\ &= 8\int_r^\infty(y-r)^2e^{-4y}\mathrm{d}y\\ &= 8\int_0^\infty z^2e^{-4z-4r}\mathrm{d}z\\ &= \exp(-4r).\end{aligned}$$

After passing to the complement, we recognize the distribution function of an exponential random variable.
d) By part a)

$$\begin{aligned}\mathbb{P}((V\wedge V')N>x,(1-(V\vee V'))N>y) &= 16\int_0^\infty\int_0^1\int_0^1 1_{[x,\infty)}(vz)1_{[y,\infty)}\big((1-v')z\big)1_{[0,v']}(v)z^2e^{-4z}\mathrm{d}v\mathrm{d}v'\mathrm{d}z\\ &= 16\int_x^\infty\int_0^1(v'-x/z)1_{[x/z,1-y/z]}(v')z^2e^{-4z}\mathrm{d}v'\mathrm{d}z\\ &= 8\int_{x+y}^\infty(1-y/z-x/z)^2z^2e^{-4z}\mathrm{d}z\\ &= 8\int_{x+y}^\infty(z-y-x)^2e^{-4z}\mathrm{d}z\\ &= 8\int_0^\infty w^2e^{-2w-4x-4y}\mathrm{d}w\\ &= e^{-4x-4y}.\end{aligned}$$

In particular, the joint pdf of $(V\wedge V')N$ and $(1-(V\vee V'))N$ is $16e^{-4x}e^{-4y}$, which is the joint pdf of two independent exponential random variables.
e) From part c), $\mathbb{E}[S_1] = \mathbb{E}[S_3] = 1/4$. Moreover, $\mathbb{E}[N] = 3/4$, since $N\sim\mathsf{Gamma}(3,4)$. Therefore,

$$\mathbb{E}[S_2] = \mathbb{E}[N-S_1-S_3] = \mathbb{E}[N]-\mathbb{E}[S_1]-\mathbb{E}[S_3] = 1/4.$$